CHEBYSHEV POLYNOMIAL APPROXIMATION TO SOLUTIONS OF THIRD ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract

Differential equations are used in variety of fields including pure and applied mathematics, engineering and physics. Many of these fields are concerned with the properties of different forms of differential equations. Solving differential equations along with certain conditions, called initial value problem (IVP) or boundary value problem, become very important in many research situations. Differential equations raised in real-world problems are not always explicitly solvable. That is, they do not always have closed from solutions. Instead, numerical methods can be used to approximate solutions. Chebyshev polynomials are two sequences of polynomials related to the sine and cosine functions. They are orthogonal polynomials that are related to De Moivre's formula. They have numerous properties which make them useful in areas like solving polynomials and approximating functions. Chebyshev approximation produces a nearly optimal approximation, coming close to minimizing the absolute error. Robertson, A. S. (2013) discussed a method for finding approximate particular solution for second order non-homogeneous ordinary differential equations. Yang Zhongshu and Zhang Hongbo (2015), in their work, developed a computational method for solving class of fractional partial differential equations with variable coefficients based on Chebyshev polynomials. In this research we developed a method to find approximate particular solution for third order linear differential equation. Here we used Chebyshev polynomial to approximate the source function and the particular solution of an ordinary differential equation. The derivatives of each Chebyshev polynomial will be represented by linear combinations of Chebyshev polynomials. Then the differential equations will become algebraic equations. Here we took first six polynomials of Chebyshev polynomials of first kind because when we approximate the function by Chebyshev polynomials, the coefficients of higher order Chebyshev polynomials are negligible. Our main objective of this study is to approximate the solution of third order linear differential equation by Chebyshev polynomial as close as possible to the exact solution. We applied our proposed method on some algebraic, trigonometry and exponential functions. This approach is compared with another well-known existing method, Euler's method. Our proposed approach provides more efficiency compared to the existing method.

Keywords: Chebyshev polynomial, Particular solution, Third order linear differential equation

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Introduction

Differential equations are used in variety of fields including pure and applied mathematics, engineering and physics. Many of these fields are concerned with the properties of different forms of differential equations. The nature and the uniqueness of solutions are the subject of pure mathematics while the systematic justification of methods for approximating solutions is the focus of applied mathematics. From celestial motion, to bridge design, to neuron interactions, differential equations are used to model virtual and physical, technological and biological process. Solving differential equations along with certain conditions, called initial value problem (IVP) or boundary value problem, become very important in many research situations.

Differential equations raised in real-world problems are not always explicitly solvable. That is, they do not always have closed from solutions. Instead, numerical methods can be used to approximate solutions. Robertson, A. S (2013) discussed a method for finding approximate particular solution for second order non-homogeneous ordinary differential equations [1].

Chebyshev polynomial is very close to the minimax polynomial which has the smallest maximum deviation from the true function. Yang Zhongshu and Zhang Hongbo (2015), in their work, developed a computational method for solving class of fractional partial differential equations with variable coefficients based on Chebyshev polynomials [3]. The main objective of our study is to approximate the solution of third order linear differential equations by Chebyshev polynomial and test its accuracy as close as possible to the exact solution. We compared our approximation method with some existing methods for three basic class of elementary functions.

Methodology

Here we are intending to approximate the particular solution numerically for the third order linear differential equation

$$ay'''(x) + by''(x) + cy'(x) + dy(x) = f(x), \tag{1}$$

where $f(x)$ is a continuous function and a, b, c, d are given constants, with the leading coefficient $a \neq 0$.

We initially express non-homogeneous term $f(x)$ of the given initial value problem as a linear combination of the Chebyshev polynomial of first kind. Then, we try to find the particular solution $y_p(x)$ in a series form which is also can be expressed in terms of the Chebyshev polynomial. For further details of Chebyshev polynomials and their properties, reader is directed to refer [1]. Now, consider the particular solution $y_p(x)$ in a series form:

$$y_p(x) = \sum_{i=0}^{k} q_i T_i(x), \tag{2}$$

where $T_i(x)$ are the Chebyshev polynomials of the first kind. The real constant coefficients q_i , $i = 0, 1, \ldots, k$ are to be determined.

Term-by-term successive differentiation for three times of the equation (2) and substitution into equation (1) yields

$$a\sum_{i=0}^{k} q_i\, T_i'''(x) + b\sum_{i=0}^{k} q_i\, T_i''(x) + c\sum_{i=0}^{k} q_i\, T_i'(x) + d\sum_{i=0}^{k} q_i T_i\,(x) = \; P_6(x)\,, \tag{3}$$

where $P_6(x)$ is the 6th degree Chebyshev polynomial approximation to the function $f(x)$. To represent $T_i'(x)$, $T_i''(x)$ and $T_i'''(x)$, we use linear combinations of Chebyshev polynomials. By equating the coefficients of $T_0, T_1, \ldots., T_6$ we can obtain a system of equations. This system of equations can be used to find the coefficients q_i , $i = 0,1,2, \ldots., 6$. To find the general solution we use initial conditions with the derivatives of equation (2).

Results and Discussion

we are intended to solve some initial value problems numerically using our proposed Chebyshev polynomial approximation method and verify its accuracy. For this purpose, we chosen examples involving polynomial, trigonometric and exponential non-homogeneous functions.

One of the commonly used approximation methods to solve initial value problems is the Euler's method. Therefore, we here compared our results with the Euler's method with the step length $h = 0.01$.

Application for an Algebraic Function

We first Consider the IVP

$$y'''(x) + 3y''(x) - 6y'(x) - 8y(x) = 4x^6 - 3x^4 + 5x^3 + 7x^2 - 1,$$

with the initial conditions $y(0) = 0,\; y'(0) = 1,\; y''(0) = -2$.

First, we approximate the polynomial function $4x^6 - 3x^4 + 5x^3 + 7x^2 - 1$ by Chebyshev polynomial,

$$4x^6 - 3x^4 + 5x^3 + 7x^2 - 1 = \frac{C_0}{2}T_0 + C_1T_1 + C_2T_2 + C_3T_3 + C_4T_4 + C_5T_5 + C_6T_6.$$

Where the coefficients $C_0, C_1, \ldots., C_6$ are calculated by comparing the coefficients.

Using the system of equations, with $n = 6$, we find the particular solution and using initial condition we find homogeneous solution.

Then general solution becomes,

$$\begin{aligned} y(x) = &-0.1642795167e^{-4x} + 310.11111111e^{-x} + 1.465277783e^{2x} \\ &- 395.537109375 + 346.289062\,x - 85.9140625(2x^2 - 1) \\ &+13.15625(4x^3 - 3x) - 1.8046875(8x^4 - 8x^2 + 1) \\ &+0.140625(16x^5 - 20x^3 + 5x) - 0.015625(32x^6 - 48x^4 + 18x^2 - 1). \end{aligned}$$

The exact solution for this initial value problem given by

$$\begin{aligned} y(x) = &-0.1642795e^{-4x} + 310.111111e^{-x} + 1.4652777778e^{2x} - 0.5x^6 + 2.25x^5 - \\ &13.6875x^4 + 49.8125x^3 - 157.671875x^2 + 307.5234375x - 311.412109375. \end{aligned}$$

The comparison of our result with the exact solution and Euler method solution is illustrated in figure 1.

Application for a Trigonometry Function

Now, let us take a look at the IVP

$$y'''(x) - 3y''(x) - y'(x) + 3y(x) = \sin 2x,$$

with the initial conditions $y(0) = 1,\ y'(0) = 0,\ y''(0) = -2.$

The non-homogeneous term $\sin 2x$ can be approximated by Chebyshev polynomial with order $n = 6$ as

$$\sin 2x = \frac{C_0}{2}T_0 + C_1T_1 + C_2T_2 + C_3T_3 + C_4T_4 + C_5T_5 + C_6T_6.$$

We use MATLAB to compute the coefficients $C_0, C_1, \ldots, C_6$.

By the system of equation (4) and by using intial conditions we obtain the particular solution and homogeneous solution respectively.

Then the general solution becomes,

$y(x) = 2.768862069e^{x} - 4.037724139e^{x} - 0.3547803443e^{3x} + 3.2860445707230496 + 8.8410650442212279x + 0.6780457786319488(2x^2 - 1) + 0.33120107847611718(4x^3 - 3x) + 0.01564362167971444(8x^4 - 8x^2 + 1) + 0.00469308650391433(16x^5 - 20x^3 + 5x).$

The exact solution of this problem is

$$y(x) = 0.175e^{-x} + 1.15e^{x} - 0.3557692308e^{3x} + 0.0461538462\sin 2x + 0.0307692308\cos 2x\ .$$

For the illustration of the comparison between our method, Euler method and exact method, see figure 2.

Application for an Exponential Function

Finally, let us consider the IVP

$$y'''(x) + 4y''(x) + y'(x) - 6y(x) = e^{x},$$

with the initial conditions $y(0) = -1,\ y'(0) = 2,\ y''(0) = 1.$

We approximate e^{x} by Chebyshev polynomial with $n = 6$:

$$e^{x} = \frac{C_0}{2}T_0 + C_1T_1 + C_2T_2 + C_3T_3 + C_4T_4 + C_5T_5 + C_6T_6.$$

We use MATLAB to compute the coefficients $C_0, C_1, \ldots, C_6$.

The general solution of the initial value problem by Chebyshev polynomial approximation is

$y(x) = 1.125041816e^{-3x} - 2.335649882e^{-2x} + 0.4995130213e^{x} - 0.363576182040806 + 0.228994710147261x - 0.076049975859307(2x^2 - 1) + 0.008353174619131(4x^3 - 3x) - 0.001386245583623(8x^4 - 8x^2 + 1) +$

$0.000075495277 67(16x^5 - 20x^3 + 5x) - 0.000007496220492(32x^6 - 48x^4 + 18x^2 - 1).$

The exact solution of this initial value problem is

$$y(x) = 1.125e^{-3x} - 2.333333e^{-2x} + 0.45833333e^{x} - 0.25e^{-x}.$$

The results for this example are compared in figure 3.

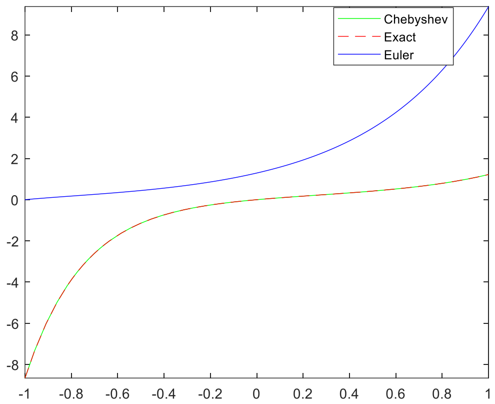

Figure 1: Comparison for polynomial function

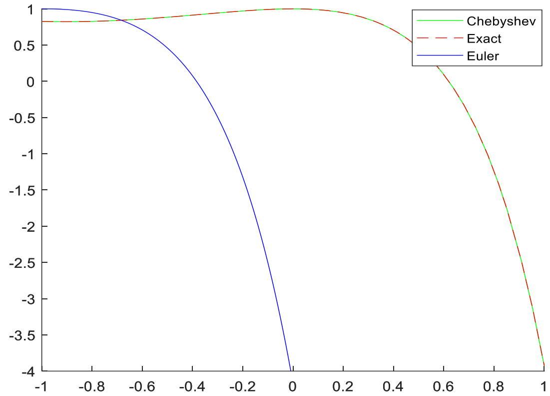

Figure 2: Comparison for trigonometric function

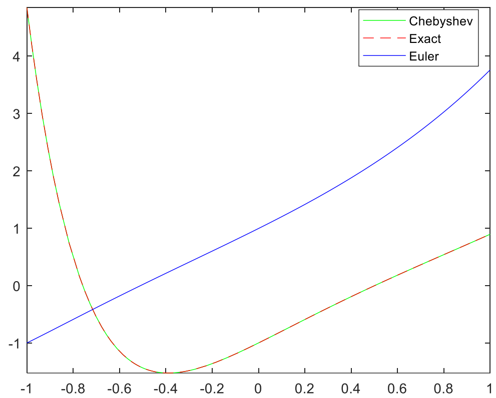

Figure 3: Comparison for exponent function

Conclusion

In this work, we developed a method to find approximate solution for third order linear differential equations using Chebyshev polynomial based on some previous work done for less order initial value problems. We used it to solve a few initial value problems with well-known elementary functions as non-homogeneous terms. We observed that our proposed method coincides with the exact solution of the initial value problem in each case.

For comparative study, we preferred Euler's method. In this comparison, we found that approximation for third order linear differential equations using Chebyshev polynomial works better than the Euler's method and it minimizes the maximum error.

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