CLASSES OF RELATIVISTIC STELLAR MODELS

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ABSTRACT: We find exact solutions to Einstein-Maxwell field equations which can be used to model the interior of charged relativistic anisotropic fluid sphere. The field equations are transformed to a simpler form using the transformation; the integration of the system is reduced to solving the condition of pressure anisotropy. The solution of the system is reduced to a difference equation with variable rational coefficients which can be solved in general. It is possible to obtain general class of solutions in terms of special functions and elementary functions for different model parameters. Our results contain particular models found previously including models of charged relativistic anisotropic spheres.

Keywords: Einstein-Maxwell Field Equations, Exact Solutions, Anisotropic Matter

INTRODUCTION

Exact solutions of the Einstein-Maxwell system are important in the description of relativistic astrophysical processes. The first exact solution to the Einstein field equations discovered was the Exterior Schwarzschild solution which describes the gravitational field outside a static spherically symmetric body. This solution is essential for a discussioin of the clasical tests of general relativity. In this paper we are concerned with anisotropic and charged fluids in general relativity theory satisfying the Einstein-Maxwell system. The gravitational filed is taken to be spherically symmetric and static since these solutions may be applied to relativistic stars. A number of researchers have examined how anisotropic matter affects critical mass, critical surface redshift and stability of highly compact bodies. These investigations are contained in the paper by Dev and Gleiser (Dev and Gleiser, 2003). Some researchers have suggested that anisotropy may be important in understanding the gravitational behavior of boson stars and the role of strange matter with densities higher than neutron stars. Mark and Harko (Mark and Harko, 2002) and Sharma and Mukherjee (Sharma and Mukherjee, 2002) suggest that anisotropy is crucial ingredient in the description of dense stars with strange matter.

The main objective of this paper is two-fold. Firstly, we seek to model a charged relativistic anisotropic sphere which is physically acceptable. Secondly, we seek to regain an uncharged and isotropic solution which satisfy the relevant physical criteria when the electric field and the anisotropic factor vanishes similar to the recent treatment of Komathiraj and Maharaj (Komathiraj and Maharaj, 2010). The approach followed in this paper has proved to be a fruitful avenue for generating new exact solutions for describing the interior spacetimes of charged anisotropic spheres. In Section 2, we obtain a simple form of the condition of pressure anisotropy extended to the electromagnetic field with the assistance of appropriate transformation. Upon specifying choices for one of the gravitational potentials, electric field and the anisotropic factor, we obtain the second order differential equation in the remaining gravitational potential which facilitates the integration procedure. It is then possible to exhibit exact solutions to the Einstein-Maxwell system in a series form. In section 3 we present two

linearly independent classes of solutions as combinations of polynomials and algebraic functions. Finally in section 4, we discuss the physical feature of the solutions.

2. THE ANISOTROPIC EQUATIONS

The metric of statics spherically symmetric spacetimes in curvature coordinates can be written as

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin\theta\, d\phi^2)$$ (1)

where λ and ν are arbitrary functions. For a charged perfect fluid the Einstein-Maxwell system of field equations is given by

$$\frac{1}{r^2}(1 - e^{-2\lambda}) + \frac{2\lambda'}{r}e^{-2\lambda} - \mu = \frac{1}{2}E^2$$ (2a)

$$-\frac{1}{r^2}(1 - e^{-2\lambda}) + \frac{2\nu'}{r}e^{-2\lambda} - p_r = -\frac{1}{2}E^2$$ (2b)

$$e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) - p_t = \frac{1}{2}E^2$$ (2c)

for the line element (1). The quantity μ is the energy density, p_r is the radial pressure, p_t is the tangential pressure and E is the electric field intensity. The Einstein-Maxwell field equations (2) describe the gravitational behaviour for an anisotropic imperfect fluid. For matter distributions with $p_r = p_t$ (isotropic pressures), the Einstein's equations for a perfect fluid may be regained from (2). We introduce the following transformation to generate new solutions. It is convenient to introduce a new coordinate x and two new metric functions $y(x)$ and $Z(x)$ defined as follows:

$$A^2 y^2(x) = e^{2\nu}, \quad Z(x) = e^{-2\lambda}, \quad x = Cr^2$$ (3)

where A and C are constants. This transformation enables us to write the system (2) to the new system

$$\frac{1 - Z}{x} - 2\dot{Z} = \frac{\mu}{C} + \frac{E^2}{2C}$$ (4a)

$$4Z\frac{\dot{y}}{y} = \frac{Z-1}{x} = \frac{p_r}{C} - \frac{E^2}{2C} \tag{4b}$$

$$4Zx^2\ddot{y} + 2x^2\dot{Z}\dot{y} + \left(\dot{Z}x - Z + 1 - \frac{E^2x}{C} - \frac{\Delta x}{C}\right)y = 0 \tag{4c}$$

$$\Delta = p_t - p_r \tag{4d}$$

where dots denote differentiations with respect to x and the quantity Δ is defined as the anisotropy factor. To integrate the system it is necessary to choose three of the variables $(\mu, p_r, p_t, Z, y, \Delta)$. In this approach we specify Z, E and Δ.

In the integration procedure, we make the following choices:

$$Z = \frac{1+kx}{1+x}\ (k \neq 1), \quad \frac{E^2}{C} = \frac{\alpha x}{(1+x)^2} \quad \text{and}$$

$$\Delta = \frac{\beta x}{(1+x)^2}$$

where k, α and β are arbitrary constants. The form chosen ensures that the metric function Z is continuous and well behaved in the interior of the star for the wide range of values of parameter k. In addition the electric field intensity and the anisotropic factor vanishes at the stellar center and has positive values in the interior of the star for relevant choices of the constants. Upon substituting these choices in equation (4c) we obtain

$$4(1+kx)(1+x)\ddot{y} + 2(k-1)\dot{y} + [(1-k)-a-\beta]y = 0 \tag{5}$$

which is the master equation of the system (4). Note that we have essentially reduced the solution of the field equation to integrating (5). The differential equation (5) has to be solved to find an exact model for a charged anisotropic fluid sphere. We now introduce a new function

$z = 1 + x$ in equation (5) to obtain

$$4z(1-k+kz)\frac{d^2Y}{dz^2} - 2(1-k)\frac{dY}{dz} + [(1-k)-a-\beta]Y = 0 \tag{6}$$

which is the second order linear differential equation in terms of the new dependent variable Y and independent variable z. With $k \neq 1$ the equation (6) can be solved using the method

of Frobenius. As the point $z = 0$ is a regular singular point of (6), there exist two linearly independent solutions of the form of a power series with centre $z = 0$. We therefore assume

$$Y = \sum_{i=0}^{\infty} a_i z^{i+b} , \ a_0 \neq 0 \tag{7}$$

where a_i are the coefficients of the series and b is a constant. For a legitimate solution we need to determine the coefficients and the constant. On substituting (7) into (6), we obtain the recurrence formula

$$a_{i+1} = \prod_{p=0}^{i} \frac{4k(p+b)(p+b-1)+(1-k-a-\beta)}{2(k-1)(p+b+1)(2p+2b-1)} a_0 \tag{8}$$

and the indicial equation

$$2a_0 b(1-k)(2b-3) = 0$$

Since $a_0 \neq 0$, we have $b = 0$ or $b = 3/2$.

Now it is possible to generate two linearly independent solutions to (6) with the assistance of (8) and (7). For the parameter value $b=0$, the first solution is given by

$$Y_1 = a_0 \left[1 + \sum_{i=0}^{\infty} \prod_{p=0}^{i} \frac{4kp(p-1)+(1-k-a-\beta)}{2(k-1)(p+1)(2p-1)} z^{i+1} \right]$$

For the parameter value $b = 3/2$ we obtain the second solution

$$Y_2 = a_0 z^{\frac{3}{2}} \left[1 + \sum_{i=0}^{\infty} \prod_{p=0}^{i} \frac{k(2p+3)(2p+1)+(1-k-a-\beta)}{(k-1)(2p+5)(2p+2)} z^{i+1} \right]$$

Since the function Y_1 and Y_2 are linearly independent we have found the general solution to (5). In terms of the original variable x, the functions Y_1 and Y_2 given above can be written and consequently the general solution become

$$y = A \left[1 + \sum_{i=0}^{\infty} \prod_{p=0}^{i} \frac{4kp(p-1)+(1-k-a-\beta)}{2(k-1)(p+1)(2p-1)} (1+x)^{i+1} \right]$$

$$+ B(1+x)^{\frac{3}{2}} \left[1 + \sum_{i=0}^{\infty} \prod_{p=0}^{i} \frac{k(2p+3)(2p+1)+(1-k-a-\beta)}{(k-1)(2p+5)(2p+2)} (1+x)^{i+1} \right] \tag{9}$$

Thus we have found the general series solution (9) to the differential equation (5). This solution is expressed in terms of a series with real arguments unlike the complex arguments given by software packages.

3. ELEMENTARY SOLUTIONS

The general solution (9) is given in the form of a series and can be expressed in terms of hypergeometric functions which are special functions. It is well known that the hypergeometric functions can be written in terms of elementary functions for particular parameter values. If we introduce the transformation

$$1 + x = KX, \ K = \frac{k-1}{k}, \ Y(X) = y(x)$$

in (5), we obtain

$$4X(1-X)\frac{d^2Y}{dX^2} - 2\frac{dY}{dX} + (K + \overline{\alpha} + \overline{\beta})Y = 0 \tag{10}$$

where $\overline{\alpha} = \dfrac{a}{k}, \ \overline{\beta} = \dfrac{\beta}{k}$.

Equation (10) is a special case of hypergeometric differential equation. It is possible to obtain two linearly independent solutions to (10) in terms of hypergeometric functions Y_1 and Y_2. These two functions are given by

$$Y_1 = F\left[-\frac{1}{2} - \frac{1}{2}\sqrt{1 + K + \overline{\alpha} + \overline{\beta}}, -\frac{1}{2} + \frac{1}{2}\sqrt{1 + K + \overline{\alpha} + \overline{\beta}}, -\frac{1}{2}, X\right] \text{ and}$$

$$Y_2 = X^{\frac{3}{2}}F\left[1 - \frac{1}{2}\sqrt{1 + K + \overline{\alpha} + \overline{\beta}}, 1 + \frac{1}{2}\sqrt{1 + K + \overline{\alpha} + \overline{\beta}}, \frac{5}{2}, X\right]$$

It is possible to expresse these two hypergeometric functions in terms of elementary functions for particular parameter values. Consequently these two functions can be written completely as combination of polynomials and algebraic functions by restricting the range of values of $\overline{\alpha}, \overline{\beta}$. Thus we can express the first category of solution to (5) as

$$y = A\left(\frac{K-1-x}{K}\right)^{\frac{1}{2}}\left[4(n+1)\sum_{i=1}^{n+1}\frac{(-4)^{i-1}(2i-1)(n+i)!}{(2i)!(n-i+1)!} \times \left(\frac{1+x}{K}\right)^i + 1\right]$$

$$B\left(\frac{1+x}{K}\right)^{\frac{3}{2}}\left[\frac{3}{n+1}\sum_{i=1}^{n}\frac{3(-4)^i(2i+2)(n+i+1)!}{(2i+3)!(n-i)!} \times \left(\frac{1+x}{K}\right)^i + 1\right] \tag{11}$$

where $K+\alpha+\beta=(2n+3)(2n+1)$.

The second category of solution is given by

$$y = A\left(\frac{K-1-x}{K}\right)^{\frac{1}{2}}\left[\frac{3}{n(n-1)}\sum_{i=1}^{n-2}\frac{(-4)^i(2i+2)(n+i)!}{(2i+3)!(n-i-2)!}\times\left(\frac{1+x}{K}\right)^i +1\right]$$

$$B\left[4n(n-1)\sum_{i=1}^{n}\frac{3(-4)^{i-1}(2i-1)(n+i-2)!}{(2i)!(n-i)!}\left(\frac{1+x}{K}\right)^i +1\right]$$

(12)

for $K+\alpha+\beta=4n(n-1)$.

Consequently we have demonstrated that elementary functions can be extracted from the general series in (9) by restricting the parameter values. It is important to observe that the Einstein-Maxwell solutions (11) and (12) apply to both isotropic and anisotropic relativistic star.

4. RESULTS AND DISCUSSION

We have found solutions to the Einstein-Maxwell system (4), by utilizing the coordinate transformation. A particular form for one of the gravitational potentials was assumed and the electric field intensity, anisotropic factor were specified. Systematic series analysis produced recurrence relation that could be solved in general. This produced new exact solutions to the Einstein-Maxwell field equation in the form of series with real arguments. For particular values of the model parameters involved it is possible to write the solution in terms of elementary functions. The anisotropic factor and the electric field intensity may vanish in the solutions and we can regain the isotropic as well as the uncharged solutions. Thus our approach has the advantage of necessarily containing a neutral isotropic stellar solution. The simple form of the solutions found facilitates the analysis of the physical features of a charged anisotropic fluid sphere. We may generate individual models found previously from our general class of solutions. These can be explicitly regained from the general series solution (9) or the elementary functions (11) and (12).

5. REFERENCES

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