

Derivation of probability density function of CIR model under a specific condition

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Abstract

In economics and finance, the Cox–Ingersoll–Ross model (or CIR model) is being generally used to model the interest rates. The CIR model is an extension of the Vasicek model and it ensures a mean reversion of the interest rates and it avoids the possibility of negative interest rates. In this work, an explicit formula for probability density function of CIR model has been derived and the final formula comport well when interest rate is close to zero.

Keywords: Probability density function, CIR model

Introduction

In economics and finance, the Cox–Ingersoll–Ross model (or CIR model) is being generally used to model the interest rates. The CIR model is an extension of the Vasicek model. Vasicek model is the first economic model to capture the value of mean reversion and CIR model ensures a mean reversion of the interest rates and it avoids the possibility of negative interest rates. In general, CIR model was established to eliminate the failures of the Vasicek model and it is given by the following stochastic differential equation (Hull & Basu, 2010):

$$dr(t) = -\theta[r(t) - \mu]dt + \sigma\sqrt{r(t)}dw, \text{ where}$$

θ = Mean reversion speed

μ = Mean reversion parameter

σ = Standard deviation that determines the volatility

w = Wiener process that models the risk factor of random market.

Probability density function is a statistical measure which corresponds to a random variable and is normally being used to calculate the probabilities or uncertainties. In

financial and economic modeling, it is often used in the creation of forecasting purposes of financial derivatives such as future exchange rates and equity prices in order to get a more complete picture regarding future market phenomena. In this work, probability density function of CIR model has been derived when interest rates are closer to zero. That is, in mathematically, $r^n \rightarrow 0$ for all $n \geq 2$ is used throughout the derivation and the final formula will give us a better yardstick for interest rate modeling.

Methodology

First and foremost, consider the CIR model: $dr(t) = -\theta[r(t) - \mu]dt + \sigma\sqrt{r(t)}dw$.

Let $X(t) = r(t) - \mu$.

Then $dX = -\theta Xdt + \sigma\sqrt{X + \mu}$. (1)

Let $y(t) = (\sqrt{X(t) + \mu})e^{\theta t}$.

Then $dy = \theta e^{\theta t} \sqrt{X + \mu} dt + \frac{e^{\theta t}}{2\sqrt{X + \mu}} dX$. (2)

Now (1) and (2) will give us $dy = \theta e^{\theta t} \left[\sqrt{X + \mu} - \frac{X}{2\sqrt{X + \mu}} \right] dt + \frac{\sigma}{2} e^{\theta t} dw$. (3)

Since $r^n \rightarrow 0$ for all $n \geq 2$, we may assume that $X^n \rightarrow 0$ for all $n \geq 2$.

Hence, we have $\left[\sqrt{X + \mu} - \frac{X}{2\sqrt{X + \mu}} \right] \approx \sqrt{\mu} \left(1 + \frac{X}{2\mu} \right) - \frac{X}{2\sqrt{\mu}} \left(1 - \frac{X}{2\mu} \right) \approx \sqrt{\mu}$.

Now (3) will give us $dy = \theta \sqrt{\mu} e^{\theta t} + \frac{\sigma}{2} e^{\theta t} dw$.

Then $\int_0^t dy = \theta \sqrt{\mu} \int_0^t e^{\theta \tau} d\tau + \frac{\sigma}{2} \int_0^t e^{\theta \tau} dw(\tau)$.

Therefore, $y(t) = y(0) + \sqrt{\mu}(e^{\theta t} - 1) + \frac{\sigma}{2} \int_0^t e^{\theta \tau} dw(\tau)$. (4)

It is well known that, $dw \sim Normal(0, d\tau)$ (Hull & Basu, 2010).

Then $\frac{\sigma}{2} \int_0^t e^{\theta \tau} dw \sim Normal \left(0, \frac{\sigma^2}{4} \int_0^t e^{2\theta \tau} d\tau \right) = Normal \left[0, \frac{\sigma^2}{8\theta} (e^{2\theta t} - 1) \right]$.

Hence, (4) implies

$$y(t) \sim \text{Normal} \left[y(0) + \sqrt{\mu} (e^{\theta t} - 1), \frac{\sigma^2}{8\theta} (e^{2\theta t} - 1) \right] = \text{Normal} \left[\sqrt{X(0) + \mu} + \sqrt{\mu} (e^{\theta t} - 1), \frac{\sigma^2}{8\theta} (e^{2\theta t} - 1) \right].$$

Then, $[\sqrt{X(t) + \mu}] \sim \text{Normal} \left[(\sqrt{X(0) + \mu})e^{-\theta t} + \sqrt{\mu}(1 - e^{-\theta t}), \frac{\sigma^2}{8\theta} (1 - e^{-2\theta t}) \right].$

Therefore, $[\sqrt{r(t)}] \sim \text{Normal} \left[(\sqrt{r(0)})e^{-\theta t} + \sqrt{\mu}(1 - e^{-\theta t}), \frac{\sigma^2}{8\theta} (1 - e^{-2\theta t}) \right].$

Now let $\Omega = \sqrt{r(0)}e^{-\theta t} + \sqrt{\mu}(1 - e^{-\theta t})$ and $\Lambda^2 = \frac{\sigma^2}{8\theta} (1 - e^{-2\theta t})$

Hence, we are getting $[\sqrt{r(t)}] \sim \text{Normal}(\Omega, \Lambda^2)$

Therefore, probability density function of $\sqrt{r(t)}$ is given by $f(\sqrt{r}) = \frac{1}{\sqrt{2\pi\Lambda^2}} e^{-\frac{(\sqrt{r}-\Omega)^2}{2\Lambda^2}}$

(Lind, Marchal & Wathen, 2008). (5)

Let F be the probability density function of $r(t)$, then it is given by

$$F(r) = \frac{1}{2\sqrt{r}} [f(\sqrt{r}) + f(-\sqrt{r})] \text{ (Teotia, 1999).} \tag{6}$$

Finally, using (5) and (6), probability density function for $r(t)$ can be determined.

Result and Discussion

Probability density function of $r(t)$ is given by $F(r) = \frac{1}{\sqrt{2\pi r\Lambda^2}} \left[e^{-\frac{(r+\Omega^2)}{2\Lambda^2}} \right] \left[\cosh\left(\frac{\Omega\sqrt{r}}{\Lambda^2}\right) \right],$

where

$$\Omega = \sqrt{r(0)}e^{-\theta t} + \sqrt{\mu}(1 - e^{-\theta t}) \text{ and } \Lambda^2 = \frac{\sigma^2}{8\theta} (1 - e^{-2\theta t})$$

This final formula can be used to determine the maximum likelihood estimators of the CIR model and hence, according to a given data set, a certain simulation can be done. Moreover, final result can be used to simulate not only interest rates but also any financial ratios which are fluctuating closer to zero such as (crude oil price/gold price).

References

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