An Application of Fourier series Expansion of a Function in a Non-Polar Spherical Coordinate System

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Abstract

Cubed sphere is one of the main tools used to avoid pole problems those arise in the selection of spherical polar coordinates. In this respect, earlier we considered a recently developed cubed sphere based on coordinate mapping over the cubed surface. The function on the sphere was treated as an ordered set of six-tuples. In that work, we established weakly orthogonal and completely orthogonal spherical harmonics of the system and developed corresponding symmetric and linear relations. Also, we found the norm of the orthogonal spherical harmonics. In this work, we explore the Fourier representation of a spherical function on this coordinate system in terms of weakly orthogonal spherical harmonics. The advantages of the linear relation between the two sets of spherical harmonics and diagonal property of the norm of the fully orthogonal spherical harmonics were in cooperated for this work. We also strength our work by giving an example to demonstrate how Fourier coefficients can be computed to represent a given spherical function in terms of the spherical harmonics of the coordinate system.

Keywords: cubed sphere, pole problems, non-polar coordinates, spherical harmonics, fourier series

Introduction

Avoiding polar singularities, which arise when using spherical polar coordinate system, gained the attraction of computational scientists in their works. The method of 'cubed sphere' became one approach to handle such problems. In approximating weather prediction models by finite difference and finite element methods, Philips, N. A. (1957), Reisswig, C. et al (2007) and Ronchi, C., Iacono, R. and Paolucci, P.S. (1996) used this method in their works. Some of the latest works in this respect can be viewed in the following articles: Archibald, M., Evans, K.J., Drake, J. and White III, J. B. (2011), Chen, C. and Xiao, F. (2008), Lauritzen, P.H., Nair, R. D. and Ullrich, P. A. (2010).

Nasir (2007) has developed a cubed sphere in this respect which has wonderful symmetric properties and defined spherical harmonics on it. Faham and Nasir (2012) developed and analyzed further in this respect. In this work, we establish Fourier series representation of a spherical function on this cubed spherical coordinate system. The work is illustrated by a simple example. The presentation of this paper is organized as follows: In next section, we recall some basic results from our previous works (Nasir (2007) and Faham and Nasir (2012)). Then, we develops the Fourier representation of a spherical function. Then, an illustrative example is given. Finally, we discuss the results and conclude some remarks.
Preliminaries, Terminologies and Notations

In this section, we brief some main results of our previous work. A cubed spherical coordinate system is defined as a six-tuple of local coordinate systems each is defined on the six faces of the cubed sphere and is given by $\varphi(r,t_1,t_2) = r\,\theta \circ \vartheta(t_1,t_2)$, where

$$\vartheta(t_1,t_2) = \{(1,t_1,t_2),(-t_1,1,t_2),(-t_2,t_1,1),(-1,-t_1,t_2),(t_1,-1,t_2),(t_2,t_1,-1)\} \tag{1}$$

maps the two dimensional faces to the cube and $\theta: \theta(P) = p/\|p\|$ maps the cube into the 2-sphere. The parameter r constitutes the standard radial coordinator. The unit sphere, (i.e. $r = 1$) is denoted by $S^2 = (X_+,Y_+,Z_+,X_-,Y_-,Z_-)$. A spherical function $f(t_1,t_2)$ is described in the cubed spherical coordinates as a six-tuple of functions

$$f(t_1,t_2) = \left\{f_{X_+}(t_1,t_2), f_{Y_+}(t_1,t_2), f_{Z_+}(t_1,t_2), f_{X_-}(t_1,t_2), f_{Y_-}(t_1,t_2), f_{Z_-}(t_1,t_2)\right\}. \tag{2}$$

A set of solutions for the eigenvalue problem

$$\Delta_S y = -l\,(l+1)y, \tag{3}$$

where l is a non-negative integer and the Laplace – Beltrami operator Δ_S , given by

$$\Delta_S = s_t\left((1+t_1^2)\frac{\partial^2}{\partial t_1^2} + 2t_1\, t_2\frac{\partial^2}{\partial t_1 \partial t_2} + (1+t_2^2)\frac{\partial^2}{\partial t_2^2} + 2t_1\frac{\partial}{\partial t_1} + 2t_2\frac{\partial}{\partial t_2}\right), \tag{4}$$

where $s_t = 1 + t_1^2 + t_2^2$, are known as the spherical harmonics. They are given by

$$y_l^{(m,n)} = \frac{p_l^{(m,n)}(t_1,t_2)}{s_t^{l/2}}, \quad m+n = l-1 \text{ or } l \tag{5}$$

where $p_l^{(m,n)}(t_1,t_2)$ are given explicitly by the non-zero real or imaginary parts of

$$p_l^{(m,n)}(t_1,t_2) = \sum_{p=m_1\ 2}^{m}\ \sum_{q=n_1\ 2}^{n} \binom{(m-p+n-q)/2}{(m-p)/2}\binom{l}{p,q}\, t_1^p\, t_2^q\, \mathbf{i}^{(l-p-q)}, \tag{6}$$

where $r_1 = r \bmod 2$, $\mathbf{i} = \sqrt{-1}$ and the subscript 2 in the summation notation indicates that the index variables increases with step 2, $\binom{a}{b}$ and $\binom{a}{b,c}$ are the binomial and trinomial coefficients respectively.

Fixing this $y_l^{(m,n)}(t_1,t_2)$ to one of the face, say X_+, of the 2-sphere, a set of continuous spherical harmonics on the 2-sphere is obtained as a six-tuple of functions

$$\mathbb{Y}(t_1,t_2) = \left(y(t_1,t_2), y\left(-\frac{1}{t_1},-\frac{t_2}{t_1}\right), y\left(-\frac{t_1}{t_2},-\frac{1}{t_2}\right), y(t_1,-t_2), y\left(-\frac{1}{t_1},\frac{t_2}{t_1}\right), y\left(\frac{t_1}{t_2},-\frac{1}{t_2}\right)\right). \tag{7}$$

The usual inner product of two functions $f(t_1,t_2)$, $h(t_1,t_2)$ is given by

$$\langle f,h\rangle_S = \sum_{F\in D}\langle f,h\rangle_F \tag{8}$$

where $\langle f,h\rangle_F = \int_{-1}^{1}\int_{-1}^{1} f_F \overline{h}_F s_t^{-3/2}\, dt_1 dt_2$ are the inner products for the functional components for one face $F \in \{X_+, Y_+, Z_+, X_-, Y_-, Z_-\}$.

The spherical harmonics $y_l^{(m,n)}$ are weakly orthogonal in the sense that they are orthogonal for distinct mode l, but are not orthogonal among the $2l+1$ functions for a mode l.

A set of completely orthogonal spherical harmonics are constructed in the form

$$Z_{l,r}^{(k)}(t_1,t_2) = \frac{Q_{l,r}^{(k)}(t_1,t_2)}{s_t^{l/2}}, \qquad r = 0,1,2,\dots,l-k, \qquad k = 0,1 \tag{9}$$

using some existing theorems with slide modifications according to our resulting differential equations. Here, polynomials $Q_{l,r}^{(k)}(t_1,t_2) = p_r(t_1)\rho^{l-r}(t_1)q_{l-r}(l;\lambda)$, where the polynomial $p_r^k(t_1)$ is given by $p_r^{(0)} + \mathbf{i}\, p_r^{(1)} = (t_1 + \mathbf{i})^r$ and the polynomials $q_{l-r}(l;\lambda)$ are given by $q_{l-r}(l;\lambda) = \sum_{n=r_1}^{l-r} a_{l,n}^{(l-r)}\lambda^n$, where $a_{l,n}^{(l-r)} = \frac{(-1)^{(l-r-n)/2}}{2^{l-r-n}}\binom{l}{(l-r-n)/2, n}$. A set of completely orthogonal spherical harmonics are then constructed very similar to the set of weakly orthogonal counter parts.

The norm of the orthogonal spherical harmonics are evaluated as

$$\left\|Z_{l,r}^{(k)}\right\|^2 = \frac{2^{2r+1}}{2l+1}\frac{(l!)^2}{(l+r)!\,(l-r)!}\begin{cases} 2\pi, & r = 0, \quad k = 0 \\ \pi, & 0 < r \le l, k = 0,1 \end{cases}. \tag{10}$$

The polynomials corresponding to weakly and completely orthogonal spherical harmonics are related by the equations

$$Q_{l,r}^{(0)} = \sum_{n=r_1\ 2}^{l-r} \frac{(-1)^{(l-r-n)/2}}{2^{(l-r-n)}}\binom{l-n}{(l-r-n)/2} P_l^{(l-n,n)} \tag{11(a)}$$

and

$$Q_{l,r}^{(1)} = \sum_{n=r_1\ 2}^{l-r} \frac{r}{l-n}\frac{(-1)^{(l-r-n)/2}}{2^{(l-r-n)}}\binom{l-n}{(l-r-n)/2} P_l^{(l-1-n,n)}. \tag{11(b)}$$

with suitable grouping, we obtain the matrix form of the linear relations as

$$\mathbb{Q}_l = \mathbb{T}_l \, \mathbb{P}_l, \tag{12}$$

where $\mathbb{T}_l = diag[A_{[(l+1)/2]}, B_{[l/2]}, C_{[l/2]}, D_{[(l-1)/2]}]$. The coefficients of the upper triangular matrices are explicitly given by $A_l(i,j) = \left(-\frac{1}{4}\right)^{j-i} \binom{2j-2}{j-i}$, $B_l(i,j) = A_l(i,j) - \frac{1}{4}A_l(i+1,j)$, $C_l(i,j) = A_l(i,j) + \frac{1}{4}A_l(i+1,j)$ and $D_l(i,j) = 2B_l(i,j) - A_l(i+1,j+1)$. The matrices, therefore, are computed by obtaining only the matrix A_l and use the above relations to obtain other matrices. The matrix A_l can be easily computed column wise by the recurrence formula

$$A_l(j,j) = 1, \;\; A_l(i-1,j) = -\frac{1}{4}\frac{(j+i-2)}{j-i+1}A_l(i,j), \;\; i = j, j-1, \cdots, 1, \;\; j = 1, 2, \cdots, n_{A,l}. \tag{13}$$

According to the regroup, we shall denote $\mathbb{Q}_l = \left[Q_{l,l}^{(0)}, Q_{l,l-1}^{(0)}, Q_{l,l-1}^{(1)}, Q_{l,l}^{(1)}\right]^T$ and $\mathbb{P}_l = \left[P_{l,l}^{(0)}, P_{l,l}^{(1)}, P_{l,l-1}^{(0)}, P_{l,l-1}^{(1)}\right]^T$. Then the set of polynomials corresponding to the weakly orthogonal spherical harmonics are as in the table 1. The linear relation among both sets of weakly and completely orthogonal spherical harmonics and the inner product of completely orthogonal spherical harmonics can be used to evaluate the inner product of weakly orthogonal spherical harmonics.

Spherical Fourier Series

Fourier series techniques could be applied to a wide array of mathematical and physical problems especially those involving linear differential equations with constant coefficients, for which the eigensolutions are sinusoids. In this work, we focus on constructing Fourier series for a spherical function defined on the surface of the cubed sphere. The Fourier series of a spherical function $f(t_1, t_2)$ can be written in terms of the weakly orthogonal spherical harmonics as

$$f(t_1, t_2) = \sum_{i=0}^{\infty} \mathbb{M}_i \, \mathbb{Y}_i \tag{14}$$

where $\mathbb{M}_i$ are row vectors of sizes $2l+1$ of coefficients given by

$$\mathbb{M}_i = \langle f, \mathbb{Y}_i \rangle \langle \mathbb{Y}_i, \mathbb{Y}_i \rangle^{-1}. \tag{15}$$

The relation between the inner products of the two sets of spherical harmonics is given in matrix form as

$$\langle \mathbb{Z}_l, \mathbb{Z}_l \rangle = \mathbb{T}_l \langle \mathbb{Y}_l, \mathbb{Y}_l \rangle \mathbb{T}_l^T \tag{16}$$

from which we obtain

Table 1: Polynomial functions corresponding to weakly orthogonal spherical harmonics

l	Polynomial group	
1	$P_{l,l}^{(0)}$	t_2
	$P_{l,l}^{(1)}$	t_1
	$P_{l,l-1}^{(0)}$	1
2	$P_{l,l}^{(0)}$	$(-1+t_2^2, -1+t_1^2)$
	$P_{l,l}^{(1)}$	$2t_1t_2$
	$P_{l,l-1}^{(0)}$	$2t_2$
	$P_{l,l-1}^{(1)}$	$2t_1$
3	$P_{l,l}^{(0)}$	$(-3t_2+t_2^3, -3t_2+3t_1^2t_2)$
	$P_{l,l}^{(1)}$	$(-3t_1+3t_1t_2^2, -3t_1+t_1^3)$
	$P_{l,l-1}^{(0)}$	$(-1+3t_2^2, -1+3t_1^2)$
	$P_{l,l-1}^{(1)}$	$6t_1t_2$
4	$P_{l,l}^{(0)}$	$(1-6t_2^2+t_2^4, 2-6t_1^2-6t_2^2+6t_1^2t_2^2, 1-6t_1^2+t_1^4)$
	$P_{l,l}^{(1)}$	$(-12t_1t_2+4t_1t_2^3, -12t_1t_2+4t_1^3t_2)$

	$P^{(0)}_{l,l-1}$	$(-4t_2+4t_2^3,-4t_2+12t_1^2t_2)$
	$P^{(1)}_{l,l-1}$	$(-4t_1+12t_1t_2^2,-4t_1+4t_1^3)$
5	$P^{(0)}_{l,l}$	$(5t_2-10t_2^3+t_2^5,10t_2-30t_1^2t_2-10t_2^3+10t_1^2t_2^3,5t_2-30t_1^2t_2+5t_1^4t_2)$
	$P^{(1)}_{l,l}$	$(5t_1-30t_1^2t_2+5t_1t_2^4,10t_1-10t_1^3-30t_1t_2^2+10t_1^3t_2^2,5t_1-10t_1^3+t_1^5)$
	$P^{(0)}_{l,l-1}$	$(1-10t_2^2+5t_2^4,2-10t_1^2-10t_2^2+30t_1^2t_2^2,1-10t_1^2+5t_1^4)$
	$P^{(1)}_{l,l-1}$	$(-20t_1t_2+20t_1t_2^3,-20t_1t_2+20t_1^3t_2)$

Also, Since $\langle \mathbb{Z}_l, \mathbb{Z}_l \rangle$ is diagonal, its inverse also diagonal with the reciprocals of the diagonal entries. Thus, the inverse matrix $\langle \mathbb{Y}_l, \mathbb{Y}_l \rangle^{-1}$ can be evaluated without inverting the matrix $\mathbb{T}_l$. Explicit expressions for the respective four blocks of $\langle \mathbb{Y}_l, \mathbb{Y}_l \rangle^{-1}$ are given by

$$\frac{2l+1}{\pi}\frac{(-1)^{i+j}}{2^{2i+2j-3}}\sum_{n=1}^{\left\lceil\frac{l+1}{2}\right\rceil}\frac{(l+2p-2)!\,(l-2p+2)!}{(l!)^2}\binom{2i-2}{i-p}\binom{2j-2}{j-p}\begin{cases}\frac{1}{2}; & p=1\\ 1; & 1<p\le\left\lceil\frac{l+1}{2}\right\rceil\end{cases},$$

$$\frac{2l+1}{\pi}\frac{(-1)^{i+j}}{2^{2i+2j-1}}\sum_{n=1}^{\left\lceil\frac{l}{2}\right\rceil}\frac{(l+2p-1)!\,(l-2p+1)!}{(l!)^2}\binom{2i-1}{i-p}\binom{2j-1}{j-p},$$

$$\frac{2l+1}{\pi}\frac{(-1)^{i+j}}{2^{2i+2j-1}}\sum_{n=1}^{\left\lceil\frac{l}{2}\right\rceil}\frac{(l+2p-1)!\,(l-2p+1)!}{(l!)^2}\frac{(2p-1)^2}{(p+i-1)(p+j-1)}\binom{2i-2}{i-p}\binom{2j-2}{j-p},$$

$$\frac{2l+1}{\pi}\frac{(-1)^{i+j}}{2^{2i+2j-1}}\sum_{n=1}^{\left\lceil\frac{l-1}{2}\right\rceil}\frac{(l+2p)!\,(l-2p)!}{(l!)^2}\frac{p^2}{(p+i)(p+j)}\binom{2i-1}{i-p}\binom{2j-1}{j-p}.$$

(18)

Application

For convenience, let $\mathbb{M}_i = \left(M_l^{(0)}, M_l^{(1)}, N_l^{(0)}, N_l^{(1)}\right)$. Then,

$$f(t_1,t_2)=\sum_{i=0}^{\infty}\mathbb{M}_i\,\mathbb{Y}_i=\sum_{l}\left(M_l^{(0)}Y_{l,l}^{(0)}+M_l^{(1)}Y_{l,l}^{(1)}+N_l^{(0)}Y_{l,l-1}^{(0)}+N_l^{(1)}Y_{l,l-1}^{(1)}\right). \quad (19)$$

Now, by equation (12), the coefficients becomes

$$f(t_1,t_2)=\sum_{i=0}^{\infty}\mathbb{M}_i\,\mathbb{Y}_i=\sum_{l}\left(M_l^{(0)}Y_{l,l}^{(0)}+M_l^{(1)}Y_{l,l}^{(1)}+N_l^{(0)}Y_{l,l-1}^{(0)}+N_l^{(1)}Y_{l,l-1}^{(1)}\right). \quad (20)$$

For a simple example, let us consider $f(t_1,t_2)=(g,0,0,0,0,0)$, where $g(t_1,t_2)$ is 1 if $0\le t_1,t_2\le 1$, takes -1 when $-1\le t_1,t_2\le 0$ and 0 otherwise on the face X_+ and zero elsewhere.

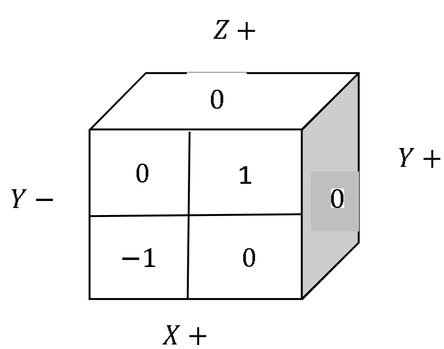

Figure 1: the function $f(t_1,t_2)$

Then, the inner product on the X_+ face is

$$\begin{aligned}\langle f, Y_{l,l}^{(k)}\rangle = \int_{-1}^{1}\int_{-1}^{1} g\, Y_l^k s_t^{-3/2} dt_1 dt_2 &= \int_{-1}^{1}\int_{-1}^{1} g\, P_l^k s_t^{-(l+3)/2} dt_1 dt_2 \\ &= \int_{0}^{1}\int_{0}^{1} P_l^k s_t^{-(l+3)/2} dt_1 dt_2 \\ &- \int_{-1}^{0}\int_{-1}^{0} P_l^k s_t^{-(l+3)/2} dt_1 dt_2.\end{aligned} \tag{21}$$

The Fourier coefficients are then computed using table 3. For example, when $l = 5$, the function $f(t_1, t_2)$ has Fourier representation

$$\begin{aligned}f(t_1,t_2) = \left(1 + t_1^2 + t_2^2\right)^{5/2}&[0.424652(5t_2 - 10t_2^3 + t_2^5) \\ &+ 0.0247504(10t_2 - 30t_1^2 t_2 - 10t_2^3 + 10t_1^2 t_2^3) \\ &- 0.0551187(5t_2 - 30t_1^2 t_2 + 5t_1^4 t_2) - 0.0551187(5t_1 - 30t_1^2 t_2 + 5t_1 t_2^4) \\ &+ 0.0247504(10t_1 - 10t_1^3 - 30t_1 t_2^2 + 10t_1^3 t_2^2) + 0.424652(5t_1 - 10t_1^3 + t_1^5)].\end{aligned} \tag{22}$$

Observations and Conclusion

In this work, we established Fourier representation of a spherical function in a non-polar cubed spherical coordinate system. The function is represented in terms of weakly orthogonal spherical harmonics. We used this representation, as an immediate application, to a simple spherical function to illustrate the advantages of computations.

Table 2: The computations of $\langle f, Y_{l,l}^{(k)}\rangle$ up to $l = 5$.

l	$\langle f, \mathbb{Y}_l^{(k)}\rangle$	values	$\langle f, \mathbb{Y}_l^{(k)}\rangle$	values
1	$\langle f, Y_{l,l}^{(0)}\rangle$	0.350188	$M_l^{(0)}$	0.350188
	$\langle f, Y_{l,l}^{(1)}\rangle$	0.350188	$M_l^{(1)}$	0.350188
	$\langle f, Y_{l,l-1}^{(0)}\rangle$	0	$N_l^{(0)}$	0
2	$\langle f, Y_{l,l}^{(0)}\rangle$	$(0,0)$	$M_l^{(0)}$	$(0,0)$
	$\langle f, Y_{l,l}^{(1)}\rangle$	0	$M_l^{(1)}$	0
	$\langle f, Y_{l,l-1}^{(0)}\rangle$	0.557909	$N_l^{(0)}$	0.52304
	$\langle f, Y_{l,l-1}^{(1)}\rangle$	0.557909	$N_l^{(1)}$	0.52304
3	$\langle f, Y_{l,l}^{(0)}\rangle$	$(-0.596819, -0.538204)$	$M_l^{(0)}$	$(-0.573505, -0.374808)$
	$\langle f, Y_{l,l}^{(1)}\rangle$	$(-0.538204, -596819)$	$M_l^{(1)}$	$(-0.374808, -0.573505)$
	$\langle f, Y_{l,l-1}^{(0)}\rangle$	$(0,0)$	$N_l^{(0)}$	$(0,0)$
	$\langle f, Y_{l,l-1}^{(1)}\rangle$	0	$N_l^{(1)}$	0
4	$\langle f, Y_{l,l}^{(0)}\rangle$	$(0,0,0)$	$M_l^{(0)}$	$(0,0,0)$
	$\langle f, Y_{l,l}^{(1)}\rangle$	$(0,0)$	$M_l^{(1)}$	$(0,0)$
	$\langle f, Y_{l,l-1}^{(0)}\rangle$	$(-0.496482, -0.325804)$	$N_l^{(0)}$	$(-0.58365, -0.0545097)$
	$\langle f, Y_{l,l-1}^{(1)}\rangle$	$(-0.325804, -0.496482)$	$N_l^{(1)}$	$(-0.0545097, -0.58365)$
5	$\langle f, Y_{l,l}^{(0)}\rangle$	$(0.323872, 0.37037, 0.0419522)$	$M_l^{(0)}$	$(0.424652, 0.0247504, -0.0551187)$
	$\langle f, Y_{l,l}^{(1)}\rangle$	$(0.0419522, 0.37037, 0.323872)$	$M_l^{(1)}$	$(-0.0551187, 0.0247504, 0.424652)$
	$\langle f, Y_{l,l-1}^{(0)}\rangle$	$(0,0,0)$	$N_l^{(0)}$	$(0,0,0)$
	$\langle f, Y_{l,l-1}^{(1)}\rangle$	$(0,0)$	$N_l^{(1)}$	$(0,0)$

For the odd mode l's, we observed that the Fourier coefficients $N_l^{(0)}$ and $N_l^{(1)}$ vanish and the values of $M_l^{(1)}$ are in the reverse order of the values of $M_l^{(0)}$. Similarly for even mode l's, we have $M_l^{(0)}$ and $M_l^{(1)}$ zeros and the values of $N_l^{(1)}$ are in the reverse order of that of $N_l^{(0)}$.Therefore, in both case, we just want to compute $\lceil l/2 \rceil$ computations although there are $2l+1$ values to be computed, here $\lceil x \rceil$ denotes the smallest integer such that $x \leq \lceil x \rceil$.

We also observe a similar relation compared to the relations defined in equation (13).

The symmetric property of $P_l^{(m,n)}(t_1, t_2)$ and $P_l^{(n,m)}(t_2, t_1)$ and the fact of the same Fourier coefficients also can be considered for efficient computations.

References

Archibald, M, Evans, KJ, Drake, J & White III, JB 2011, 'Multi-wavelet Discontinuous Galerkin-Accelerated Exact Linear Part (ELP) Method for the Shallow-Water Equations on the Cubed Sphere', *Monthly Weather Review*, vol. 139(2), pp. 457 – 473.

Chen, C & Xiao, F 2008, 'Shallow water model on cubed-sphere by multi-moment finite volume method', *Journal of Computational Physics* vol. 227(10), pp. 5019 – 5044.

Faham MAAM & Nasir HM 2012a, 'Weakly orthogonal spherical harmonics in non-polar spherical coordinates and its applications to functions on cubed sphere', *SQU Journal for science*, vol. 17 (2), pp. 200 – 213.

Faham MAAM & Nasir HM 2012b, 'Relational properties of weakly orthogonal and orthogonal spherical harmonics. *Annual Science Research Session* (ASRS), South Eastern University of Sri Lanka.

Lauritzen, PH, Nair, RD & Ullrich, PA 2010, 'A conservative semi-Lagrangian multi-tracer transport scheme (CSLAM) on the cubed-sphere grid', *Journal of Computational Physics* vol. 229(5), pp. 1401 – 1424.

Nasir, HM 2007, 'Spherical harmonics in a non-polar coordinate system and application to Fourier series in 2-sphere', *Mathematical Methods in the Applied Sciences* vol. 30 (14), pp. 1843 – 1854.

Phillips, NA 1957, 'A map projection system suitable for large-scale numerical weather prediction', in Syono, S., ed., 75th Anniversary Volume. *Journal of Meteorological Society of Japan*, vol. 35, pp. 262 – 267.

Reisswig, C, Bishop, NT, Lai, CW, Thornburg, J, & Szilágyi, B 2007, 'Characteristic evolutions in numerical relativity using six angular patches', *Classical and Quantum Gravity* vol. 24, pp. 237 – 339.

Ronchi, C, Iacono, R & Paolucci, PS 1996, 'The 'Cubed Sphere': A New Method for the Solution of Partial Differential Equations in Spherical Geometry', *Journal of Computational Physics*, vol. 124, pp. 93 – 114.